We begin with the following Theorem of Makarov.

Thm (Makarov). If Ω is s.c. domain, $\dim \omega = \underline{\dim}\, \omega = 1$.

Remarks: 1) $\operatorname{supp} \omega = \partial\Omega$, while $\dim \operatorname{supp} \omega$ can be 2.

2) Not true in higher dimension:
Thm (Wolff) $\exists\ \Omega \subset \mathbb{R}^n, n \geq 3$: $\dim \omega > n-1$
Conjecture. In $\mathbb{R}^n$, $\sup \dim \omega = n-1+\frac{n-2}{n-1}$.

3) $\dim = \underline{\dim} = 1 \iff \omega$-a.e. $\dim_\omega(x) = 1$.

Step 1. $\dim \omega \leq 1$. Moreover, if $\frac{h(t)}{t} \xrightarrow[t\to 0]{} 0$, then $\exists K \subset \partial\Omega$: $\omega(K) = 1$, $H_h(K) = 0$.

Let $\varphi: \mathbb{D} \to \Omega$, $\varphi(0) = z_0$ - Conformal

Def. A sequence of points $z_k \in \mathbb{D}$ is non-tangentially dense on $A \subset \mathbb{T}$ if $\forall\ \zeta \in A\ \exists\ z_{n_k} \to \zeta$ non-tangentially, i.e. within an angle.

Lemma (Makarov). Let (z_n) be non-tangentially dense on A, $w_n := \varphi(z_n)$, $r_n := \operatorname{dist}(w_n, \partial\Omega)$, $B_n := B(w_n, 2r_n)$, $V := \partial\Omega \cap (\cup B_n)$. Then $m_1(A \setminus \varphi^{-1}(V)) = 0$.

Pf (of lemma). Let $m_1(A \setminus \varphi^{-1}(V)) > 0$.
Let W_k be the component of $B_k \cap \Omega$ containing w_k.
$\partial\Omega$ - connected, so $\omega_{w_k}(V, w_k) \geq C$ - some absolute constant

$\left(\quad \omega_{w_n}(V, w_n) \overset{\text{reflection}}{\geq} \frac{1}{2}\, \omega_{w_n}(V \cup \bar{V}, w_n) \geq \omega_{w_n}(\quad) = \text{Const} \right)$ (radius)

$\&$ $u(z) := \omega_{\varphi(z)}(V, \Omega)$ - harmonic in z, $u(z_k) \geq c$.

$\overset{\text{Conf. invariance}}{=} \omega_z(\varphi^{-1}(V), \mathbb{D})$. $\underline{\lim}_{z \overset{\angle}{\to} \zeta} u(z) \geq c$ a.e. on A.

On the other hand, $\lim_{z \overset{\angle}{\to} \zeta} u(z) = \lim_{z \overset{\angle}{\to} \zeta} (\varphi^{-1}(V), \mathbb{D}) = \chi_{\varphi^{-1}(V)}$,

so, a.e. on A, $\chi_{\varphi^{-1}(V)} \geq c$ ∎

Thm (Privalov).
Let g be a non-constant meromorphic function on $\mathbb{D}$.
Then a.e. $\zeta \in S^1$, $\underline{\lim}_{z \overset{\angle}{\to} \zeta} g(z) < \infty$.

Apply it to φ'.
Let $E_n := \{\zeta \in S^1 : \underline{\lim}_{z \overset{\angle}{\to} \zeta} |\varphi'(z)| < n\}$.
$E_n \uparrow$, $m_1(\cup E_n) = 1$.

Let $I(z) := \{\zeta \in \partial\mathbb{D} : |z - \zeta| < 2(1-|z|)\}$ - arc.
z is in cone from $\zeta \iff \zeta \in I(z)$
Thus $\forall\ \zeta \in E_n\ \exists$ arbitrary small $I(z)$: $\zeta \in I(z)$ and
1) $|\varphi'(z)| < n$
2) $1 - |z|^2 < \delta_n$, where δ_n chosen so that $t < \delta_n \Rightarrow$
$\frac{h(t)}{t} < \frac{\varepsilon}{n 2^{n+2}}$.

By Vitali covering lemma can select $I(z_{n,j})$ non-intersecting, such that
1) $|\varphi'(z_{n,j})| < n$
2) $1 - |z_{n,j}|^2 < \delta_n$
3) $|E_n \setminus \cup I(z_{n,j})| = 0$

Then $\sum_j h(2|\varphi'(z_{n,j})|(1-|z_{n,j}|^2)) \le C\sum_j \frac{\varepsilon}{n 2^{n+2}} n\,|I(z_{n,j})|$

$\le \frac{C\varepsilon}{2^n}$.

Fix $\varepsilon > 0$. Take $(z_k) = (z_{n,j})_{n,j}$ – non-tangentially dense on $\cup E_n$. $w_k := f(z_k)$, $r_k := \mathrm{dist}(w_k, \partial\Omega)$.

So $\varphi(\cup E_n)$ has full harmonic measure, covered a.e. by $\cup B(w, 2r_k)$. By Shwarz' Lemma,

$2r_k \le 2|\varphi'(z_k)|(1-|z_k|^2) \le 4\varepsilon$, and

$\sum h(2r_k) \le C\varepsilon$. So $m_{h,4\varepsilon} \le C\varepsilon$ ∎

Step 2. $\dim\omega \ge 1$. Moreover, $\exists C > 0$: $h(t) := t\exp\left(C\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}\right)$ such that $K \subset \partial\Omega$, $\omega(K) > 0 \Rightarrow m_h(K) > 0$.

("Moreover", since $\forall \alpha < 1$, $\lim \frac{h(t)}{t^\alpha} = 0$).

Lemma (Rohde) Let $0 < \delta < \varepsilon$, $\frac{1}{2} \le r < 1$, $A \subset \mathbb{T}$.

If 1) $|\varphi(A)| \le \varepsilon$.

2) $\forall \zeta \in A$: $|\varphi(r\zeta) - \varphi(\zeta)| \le \varepsilon$

3) $(1-r)|\varphi'(r\zeta)| \ge \delta$ $\forall \zeta \in A$.

Then A can be covered by $\le C_1\left(\frac{\varepsilon}{\delta}\right)^2$ sets of diameter $\le 1-r$.

Pf Take small c. Take dyadic squares of size $c\delta$,

Let $(Q_k)_{k=1}^m = \{$dyadic squares : $\varphi(rA) \cap Q_k\} \ne \emptyset$.

By 1)+2), $|\varphi(rA)| \le 3\varepsilon$. So

$\mathrm{Area}(Q_1 \cup Q_2 \ldots \cup Q_m) = (c\delta)^2 m \le C_2\varepsilon^2$, so

$m \le c_1\left(\frac{\varepsilon}{\delta}\right)^2$.

Let $A_k := \{\zeta \in A : f(r\zeta) \in Q_k\}$. Check that $|A_k| \le 1-r$.

Assume $|A_k| > 1-r$: $\exists \zeta, \zeta' \in A_k$: $|r\zeta - r\zeta'| > 1-r$ $\Rightarrow$

(by 3) $\delta < (1-r^2)|\varphi'(r\zeta)| \overset{\text{distortion}}{\le} c_\delta |f(r\zeta) - f(r\zeta')| \le 2C_5|Q_k| <$

$2c_5 c\delta$. Take now $c < \frac{1}{2c_5}$ to get contradiction ∎

Thm (Makarov's LIL): $\exists c > 0$ – absolute:

A.e. $\zeta \in \mathbb{T}$: $\overline{\lim_{r\to 1-}} \frac{|\log f'(r\zeta)|}{\sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}}} \le c$.

Will prove later – the proof is probabilistic

Assume LIL. Find $A' \subset \varphi^{-1}(K)$ with $m_1(A') > 0$.

and

$\left|\log|\varphi'(r\zeta)|\right| \le \psi(r) := C\sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}}$ for

$\forall r > r_0$ and $\forall \zeta \in A'$.

Easy to see, by integration, that $|\varphi(\zeta) - \varphi(r\zeta)| = 2(1-r)e^{\psi(r)}$.

Let B_k – open cover of $\varphi(A') \subset K$.

$A_k := \varphi^{-1}(B_k)$, $\varepsilon_k := |B_k|$.

Define r_k by $\varepsilon_k = (1-r_k)\exp(\psi(r_k))$.

$\delta_k := (1-r_k)\exp(-\psi(r_k))$.

By Rohde's Lemma, A_k can be covered by $\leq C\left(\frac{\varepsilon_k}{\delta_k}\right)$ sets of diameter $(1-r_k)$. So
$m_1(A') \leq \sum m_1(A_k) \lesssim \sum (1-r_k)\exp(4\psi(r_k)) \lesssim \sum h(\varepsilon_k)$.
So $m_h(K) \geq m_h(\varphi(A')) \gtrsim m_1(A') > 0$ ∎

In fact, the same methods allow us to perform multifractal analysis of harmonic measure.

To this end, define *packing spectrum* of a measure by

$$\pi(t) := \sup\{q : \forall \delta > 0\ \exists B(z_j, \delta_j) : \sum \delta_j^t \mu(B(z_j,\delta))^q \geq 1,\ \delta_j \leq \delta,\ B(z_k,\delta_k) \cap B(z_j,\delta_j) = \emptyset,\ j \neq k\} =$$

$$= \overline{\lim_{\varepsilon \to 0}} \frac{\log L(t,\varepsilon)}{\log \frac{1}{\varepsilon}}, \text{ where } L(t,\varepsilon) = \sup\{\sum \delta(B_j)^t : B_j \cap B_k = \emptyset,\ \mu(B_j) = \varepsilon\},$$

and *dimension packing spectrum* as

$$\tilde{f}(\alpha) := \lim_{\eta \to 0+} \overline{\lim_{\delta \to 0}} \frac{\log P(\delta, \alpha, \eta)}{\log \frac{1}{\delta}}, \text{ where}$$

Plays the same role as f, but for upper Minkowski dimension.

$$P(\delta; \alpha, \eta) := \max \#\{\text{disjoint discs } B = B(z,\delta) \text{ with } \delta^{\alpha+\eta} \leq \mu B \leq \delta^{\alpha-\eta}\}.$$

Then it is an easy exercise to see that $f(\alpha) \leq \tilde{f}(\alpha)$.

Also, if in $L(t,\varepsilon)$ you only sum over the disks from $P(\delta; \alpha, \eta)$, you get that $L(t,\varepsilon) \geq \left(\frac{1}{\varepsilon}\right)^{\frac{\tilde{f}(\alpha)}{\alpha}} \cdot \varepsilon^{\frac{t}{\alpha}}$, so

$\pi(t) \geq \sup_{\alpha > 0} \frac{\tilde{f}(\alpha) - t}{\alpha}$. In fact, one can prove that there is equality!

So $f(\alpha) \leq \tilde{f}(\alpha) \leq \inf(\alpha\pi(t) + t)$.

The conformal maps counterpart is the *integral spectrum*

$$\beta(t) := \overline{\lim_{r \to 1}} \frac{\log \int |\varphi'(r\zeta)|^t |d\zeta|}{\log \frac{1}{1-r}}$$

Then

Thm (Makarov) 1). $\beta(t) \geq \pi(t) - t + 1$

2) $\beta(t) = \pi(t) - t + 1$ for $t \leq t_* := \sup_\alpha f(\alpha)$.

Now, let us consider *the Universal spectra*:

$$B(t) := \sup_{\varphi - \text{bounded conformal}} \beta_\varphi(t), \quad F(\alpha) = \sup_{\Omega\text{-s.c., bounded}} f(\alpha), \quad \tilde{F}(\alpha) := \sup_{\Omega\text{-s.c. bounded}} \tilde{f}(\alpha), \quad \Pi(t) := \sup_{\Omega\text{-s.c. bounded}} \pi(t)$$

Thm (Makarov). $B(t) = \Pi(t) - t + 1$, $F(\alpha) = \tilde{F}(\alpha) = \inf_t(\alpha\Pi(t) + t)$,

$$\Pi(t) = \sup_\alpha\left(\frac{F(\alpha) - t}{\alpha}\right).$$

$$\Pi(t) = \sup_{\alpha}\left(\frac{F(\alpha)-t}{\alpha}\right).$$

The key to the proof : Fractal Approximation.

Thm (Makarov) $B(t) = \sup_{CER} \beta(t)$.

Conjecture (Kraetser). $B(t) = \begin{cases} t^2/4, & |t| \le 2 \\ |t|-1, & |t| \ge 2. \end{cases}$ $F(\alpha) = 2 - \frac{1}{\alpha}$, $\alpha \le \frac{1}{2}$.

Known: $B'(0) = 0$, $B(t) = |t|-1$ for $t \ge 2$.

$B(-2) = 1$ - Brennan's conjecture.

$B(1) = \frac{1}{4}$ - Carleson-Jones conjecture

What about general domains?

Thm (Jones-Wolff) $\dim \omega \le 1$ $\forall \Omega \subset \mathbb{C}$.

Multifractal Analysis: $\tilde{F}(\alpha) = \alpha$ - nothing interesting, but

Thm (Jones-Makarov-Smirnov-B) $F(\alpha) = \begin{cases} \alpha, & \alpha \le 1 \\ F_{SC}(\alpha), & \alpha \ge 1. \end{cases}$

The proof consists of two theorems.

Thm (Makarov-Smirnov-B) Let Ω be the basin of attraction of ∞ of some hyperbolic polynomial. Then $f(\alpha) \le F_{SC}(\alpha)$ (Actually, need all critical pts to escape to ∞ - polynomial Cantor sets).

Thm (Jones-B) $F(\alpha) = \sup_{\text{polynomial Cantor sets}} f(\alpha)$